KOLEKSI PEMBUKTIAN: MATEMATIK TAMBAHAN SPM

PEMBUKTIAN DAN PENERBITAN RUMUS SERTA KONSEP TERPILIH

CIKGU MUHAMMAD FAZDHLY BIN ABDUL MUTTALIB

Guru Cemerlang Matematik Tambahan

PRAKATA

Koleksi pembuktian ini dikumpulkan bagi membantu guru-guru dan murid-murid untuk mendapatkan idea serta panduan bagaimanakah suatu rumus tertentu dalam Matematik Tambahan diterbitkan.

Kesemua hasil penulisan pembuktian adalah berdasarkan

1. Pengetahuan penulis sendiri.
2. Rujukan terhadap beberapa buku antarabangsa.
3. Rujukan terhadap beberapa buku tempatan termasuklah buku terbitan IPTA, STPM dan buku teks sekolah menengah (KBSM dan KSSM).

Dalam pengumpulan pembuktian ini, saya turut disokong oleh rakan-rakan yang membantu membuat semakan dan cadangan penambahbaikan iaitu

1) En Ku Haslizam bin Ku Azmi (GC Matematik)
2) En Haris Fadzli bin Awang (admin AMSG)
3) En Aizuddin bin Yusoff (admin AMSG)
4) En Noor Ishak bin Mohd Salleh (admin AMSG)
5) En Clemente Chock (admin AMSG)

Koleksi ini diberikan secara percuma sebagai amal jariah. Saya Cuma berharap agar pengguna mendoakan anak murid saya mendapat keputusan yang cemerlang untuk subjek Matematik Tambahan dalam peperiksaan SPM.

Sekiranya terdapat kesilapan dalam penulisan, saya dengan rendah hati memohon kemaafan dan mengalu-alukan cadangan penambahbaikan atau pembetulan.

First release: 28 Jun 2021

Cikgu Fazdhly

Ipoh, Perak

TINGKATAN 4

KONSEP FUNGSI SONGSANG

Diberi bahawa $f(x) = x^2, x \geq 0$ dan $g(x) = -\sqrt{x}, x \geq 0$. Cari $fg(x)$ dan $gf(x)$. Seterusnya, tentukan sama ada $g(x)$ ialah fungsi songsang bagi $f(x)$ atau tidak. Nyatakan alasan anda.

Given that $f(x) = x^2, x \geq 0$ and $g(x) = -\sqrt{x}, x \geq 0$. Find $fg(x)$ and *$gf(x)$. Hence, determine whether $g(x)$ is the inverse function of $f(x)$ or not. State your reason.*

Penyelesaian / *Solution*:

Fungsi gubahan $f\,g(x)$
Composite function $f\,g(x)$

$$f\,g(x) = f\left[g(x)\right]$$
$$= \left(-\sqrt{x}\right)^2$$
$$f\,g(x) = x$$

Fungsi gubahan $g\,f(x)$
Composite function $g\,f(x)$

$$g\,f(x) = g\left[f(x)\right]$$
$$= -\sqrt{x^2}$$
$$g\,f(x) = -x$$

Kerana $g\,f(x) \neq x$, maka $g(x)$ bukan fungsi songsang bagi $f(x)$. ■

Because $g\,f(x) \neq x$, then $g(x)$ is not the inverse function of $f(x)$. ■

MENERBITKAN RUMUS KUADRATIK

Diberi persamaan kuadratik $ax^2 + bx + c = 0$, dengan keadaan a, b dan c ialah pemalar, $a \neq 0$. Tunjukkan bahawa punca-punca bagi persamaan kuadratik ini ialah
Given the quadratic equation, $ax^2 + bx + c = 0$, where a, b and c are constants, $a \neq 0$. Show that the roots of the quadratic equation is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Penyelesaian / *Solution*:

$$ax^2 + bx + c = 0$$

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Penyempurnaan Kuasa Dua
Completing the Square

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \blacksquare$$

HASIL TAMBAH PUNCA & HASIL DARAB PUNCA

Diberi persamaan kuadratik $ax^2 + bx + c = 0$, dengan keadaan a, b dan c ialah pemalar, $a \neq 0$ mempunyai punca-punca α dan β. Tunjukkan bahawa

$$\alpha + \beta = -\frac{b}{a} \text{ dan } \alpha\beta = \frac{c}{a}$$

Given the quadratic equation, $ax^2 + bx + c = 0$, where a, b and c are constants, $a \neq 0$ has roots α and β. Show that

$$\alpha + \beta = -\frac{b}{a} \textit{ and } \alpha\beta = \frac{c}{a}$$

Penyelesaian / *Solution*:

$$ax^2 + bx + c = 0$$

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad \ldots①$$

$$(x - \alpha)(x - \beta) = 0$$

$$x^2 - \alpha x - \beta x + \alpha\beta = 0$$

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0 \quad \ldots②$$

Bandingkan ① dan ②

$$\alpha + \beta = -\frac{b}{a} \qquad \text{dan} \qquad \alpha\beta = \frac{c}{a} \qquad \blacksquare$$

LOGARITMA: HUKUM HASIL DARAB

Diberi $m = a^x$ dan $n = a^y$, tunjukkan bahawa

$$\log_a mn = \log_a m + \log_a n$$

Given that $m = a^x$ *and* $n = a^y$, *show that*

$$\log_a mn = \log_a m + \log_a n$$

Penyelesaian / *Solution*:

Jika $m = a^x$, maka $\log_a m = x$

Jika $n = a^y$, maka $\log_a n = y$

Takrifan logaritma (asas a)
Definition of logarithm (*base* a)

$$mn = a^x \times a^y$$

$$mn = a^{x+y}$$

$$\log_a mn = x + y$$

$$\log_a mn = \log_a m + \log_a n \qquad \blacksquare$$

LOGARITMA: HUKUM HASIL BAHAGI

Diberi $m = a^x$ dan $n = a^y$, tunjukkan bahawa

$$\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$$

Given that $m = a^x$ *and* $n = a^y$, *show that*

$$\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$$

Penyelesaian / *Solution*:

Jika $m = a^x$, maka $\log_a m = x$

Jika $n = a^y$, maka $\log_a n = y$

$$\frac{m}{n} = \frac{a^x}{a^y}$$

$$\frac{m}{n} = a^{x-y}$$

$$\log_a \left(\frac{m}{n}\right) = x - y$$

Takrifan logaritma (asas a)
Definition of logarithm (*base* a)

$$\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n \qquad \blacksquare$$

LOGARITMA: PENUKARAN ASAS

Jika a, b dan c adalah nombor-nombor positif, $a \neq 1$ dan $c \neq 1$, tunjukkan kaedah untuk menerbitkan
If a, b and c are positive numbers, $a \neq 1$ and $c \neq 1$, show the method to derive

$$\log_a b = \frac{\log_c b}{\log_c a}$$

Penyelesaian / *Solution*:

Andaikan $\log_a b = x$, oleh itu $b = a^x$

$$a^x = b$$

$$\log_c a^x = \log_c b$$

$$x \log_c a = \log_c b$$

$$x = \frac{\log_c b}{\log_c a}$$

$$\log_a b = \frac{\log_c b}{\log_c a} \qquad \blacksquare$$

Ambil $\log_c$ di kedua-dua belah
Take $\log_c$ to both sides

RUMUS S_n BAGI JANJANG ARITMETIK

Jika a ialah sebutan pertama dan d ialah beza sepunya bagi suatu janjang aritmetik, tunjukkan bahawa hasil tambah n sebutan pertama bagi janjang itu ialah
If a is the first term and d is the common difference of an arithmetic progression, show that the sum of the first n terms of the progression is

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Penyelesaian / *Solution*:

$$\begin{array}{rcccccccc}
 & S_n = & T_1 & + & T_2 & + & T_3 & +\ldots+ & T_n \\
 & S_n = & a & + & [a+d] & + & [a+2d] & +\ldots+ & [a+(n-1)d] \\
+ & S_n = & [a+(n-1)d] & + & [a+(n-2)d] & + & [a+(n-3)d] & +\ldots+ & a \\
\hline
 & 2S_n = & 2a+(n-1)d & + & 2a+(n-1)d & + & 2a+(n-1)d & +\ldots+ & 2a+(n-1)d
\end{array}$$

$$2S_n = n[2a+(n-1)d]$$

$$S_n = \frac{n}{2}[2a+(n-1)d] \quad \blacksquare$$

Tulis hasil tambah dalam tertib sebutan menurun
Write the sum in descending order of terms

RUMUS S_n BAGI JANJANG ARITMETIK

Dalam satu janjang aritmetik dengan n bilangan sebutan, diberi bahawa a ialah sebutan pertama, l ialah sebutan terakhir dan d ialah beza sepunya. Jika S_n ialah hasil tambah n sebutan pertama bagi janjang itu, tunjukkan

In an arithmetic progression with n number of term, it is given that a is the first term, l is the last term and d is the common difference. If S_n is the sum of the first n terms of the progression, show that

$$S_n = \frac{n}{2}[a + l]$$

Penyelesaian / *Solution*:

$$\begin{array}{rcccccccc}
 & S_n = & T_1 & + & T_2 & + & T_3 & +\ldots+ & T_n \\
 & S_n = & a & + & [a+d] & + & [a+2d] & +\ldots+ & [a+(n-1)d] \\
+ & S_n = & [a+(n-1)d] & + & [a+(n-2)d] & + & [a+(n-3)d] & +\ldots+ & a \\
\hline
 & 2S_n = & 2a+(n-1)d & + & 2a+(n-1)d & + & 2a+(n-1)d & +\ldots+ & 2a+(n-1)d
\end{array}$$

$$2S_n = n[2a+(n-1)d]$$

$$S_n = \frac{n}{2}[a+a+(n-1)d]$$

$$S_n = \frac{n}{2}[a+T_n]$$ tetapi $T_n = l$ (sebutan terakhir dalam janjang itu)

$$S_n = \frac{n}{2}[a+l]$$ ■

RUMUS S_n BAGI JANJANG GEOMETRI

Jika a ialah sebutan pertama dan r ialah nisbah sepunya bagi suatu janjang geometri, tunjukkan bahawa hasil tambah n sebutan pertama bagi janjang itu ialah
If a is the first term and r is the common ratio of a geometric progression, show that the sum of the first n terms of the progression is

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

Penyelesaian / *Solution*:

Darab r pada setiap sebutan
Multiply r to each term

$$\begin{aligned} S_n &= T_1 + T_2 + T_3 + \dots + T_n \\ S_n &= a + ar + ar^2 + \dots + ar^{n-1} \quad \dots① \\ rS_n &= ar + ar^2 + ar^3 + \dots + ar^n \quad \dots② \end{aligned}$$

② – ①:

$$rS_n - S_n = \left(ar + ar^2 + ar^3 + \dots + ar^n\right) - \left(a + ar + ar^2 + \dots + ar^{n-1}\right)$$

$$S_n(r-1) = ar^n - a$$

$$S_n(r-1) = a\left(r^n - 1\right)$$

$$S_n = \frac{a\left(r^n - 1\right)}{r-1} \quad \blacksquare$$

RUMUS S_n BAGI JANJANG GEOMETRI

Jika a ialah sebutan pertama dan r ialah nisbah sepunya bagi suatu janjang geometri, tunjukkan bahawa hasil tambah n sebutan pertama bagi janjang itu ialah
If a is the first term and r is the common ratio of a geometric progression, show that the sum of the first n terms of the progression is

$$S_n = \frac{a(1-r^n)}{1-r}$$

Penyelesaian / *Solution*:

Darab r pada setiap sebutan
Multiply r to each term

$$\begin{aligned} S_n &= T_1 + T_2 + T_3 + \dots + T_n \\ S_n &= a + ar + ar^2 + \dots + ar^{n-1} \qquad \dots① \\ rS_n &= ar + ar^2 + ar^3 + \dots + ar^n \qquad \dots② \end{aligned}$$

① – ②:

$$S_n - rS_n = \left(a + ar + ar^2 + \dots + ar^{n-1}\right) - \left(ar + ar^2 + ar^3 + \dots + ar^n\right)$$

$$S_n(1-r) = a - ar^n$$

$$S_n(1-r) = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{1-r} \qquad \blacksquare$$

RUMUS S_∞ BAGI JANJANG GEOMETRI

Jika a ialah sebutan pertama dan r ialah nisbah sepunya bagi suatu janjang geometri, tunjukkan bahawa hasil tambah ketakterhinggaan bagi janjang itu ialah
If a is the first term and r is the common ratio of a geometric progression, show that the sum to infinity of the progression is

$$S_\infty = \frac{a}{1-r}, |r| < 1$$

Penyelesaian / *Solution*:

Contoh: $0.3^{99999} \approx 0$
Example: $0.3^{99999} \approx 0$

Pertimbangkan $S_n = \dfrac{a\left(1-r^n\right)}{1-r}, |r| < 1$

Apabila $r \to \infty, r^n \to 0,$ oleh itu $1-r^n \to 1$

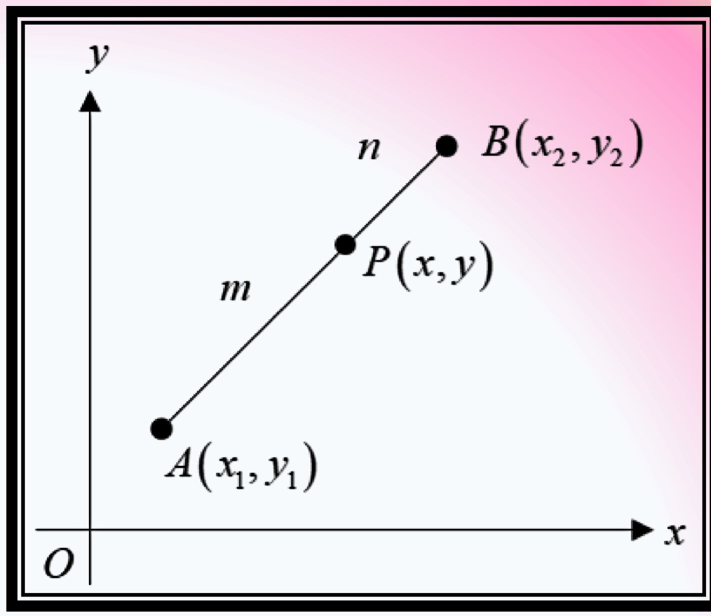
Maka, $S_\infty = \dfrac{a}{1-r}, |r| < 1$ ■

RUMUS TITIK PEMBAHAGI TEMBERENG GARIS DALAM NISBAH *m* : *n*

Dalam rajah berikut, $P(x, y)$ membahagi tembereng garis AB dalam nisbah $m : n$. Tunjukkan bahawa

In the following diagram, $P(x, y)$ divides the line segment AB in the ratio $m : n$. Show that

$$(x, y) = \left(\frac{nx_1 + mx_2}{m+n}, \frac{ny_1 + my_2}{m+n}\right)$$



Penyelesaian / *Solution*:

$$\frac{x - x_1}{x_2 - x} = \frac{m}{n}$$

$$nx - nx_1 = mx_2 - mx$$

$$nx + mx = mx_2 + nx_1$$

$$x(m+n) = nx_1 + mx_2$$

$$x = \frac{nx_1 + mx_2}{m+n}$$

$$\frac{y - y_1}{y_2 - y} = \frac{m}{n}$$

$$ny - ny_1 = my_2 - my$$

$$ny + my = my_2 + ny_1$$

$$y(m+n) = ny_1 + my_2$$

$$y = \frac{ny_1 + my_2}{m+n}$$

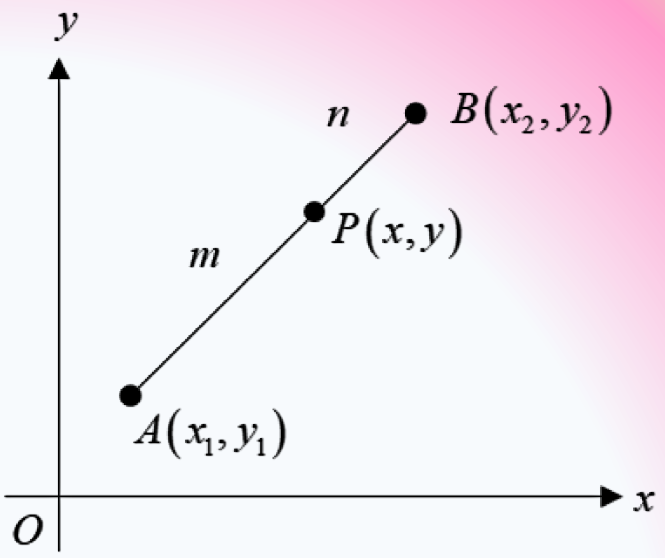
Oleh itu, $(x, y) = \left(\frac{nx_1 + mx_2}{m+n}, \frac{ny_1 + my_2}{m+n}\right)$ ■

RUMUS TITIK PEMBAHAGI TEMBERENG GARIS DALAM NISBAH *m* : *n*

Dalam rajah berikut, $P(x, y)$ membahagi tembereng garis AB dalam nisbah $m : n$. Tunjukkan bahawa

In the following diagram, $P(x, y)$ divides the line segment AB in the ratio $m : n$. Show that

$$(x, y) = \left(\frac{nx_1 + mx_2}{m+n}, \frac{ny_1 + my_2}{m+n}\right)$$



Penyelesaian / *Solution*:

KAEDAH ALTERNATIF
ALTERNATIVE METHOD

$$x = x_1 + \frac{m}{m+n}(x_2 - x_1)$$

$$x = \frac{x_1(m+n) + m(x_2 - x_1)}{m+n}$$

$$x = \frac{mx_1 + nx_1 + mx_2 - mx_1}{m+n}$$

$$x = \frac{nx_1 + mx_2}{m+n}$$

$$y = y_1 + \frac{m}{m+n}(y_2 - y_1)$$

$$y = \frac{y_1(m+n) + m(y_2 - y_1)}{m+n}$$

$$y = \frac{my_1 + ny_1 + my_2 - my_1}{m+n}$$

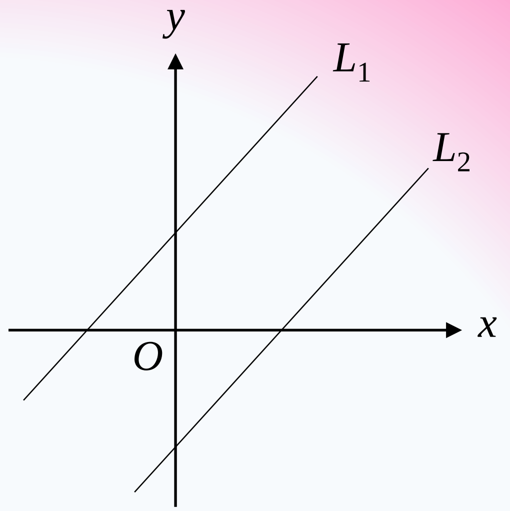
$$y = \frac{ny_1 + my_2}{m+n}$$

Oleh itu, $(x, y) = \left(\dfrac{nx_1 + mx_2}{m+n}, \dfrac{ny_1 + my_2}{m+n}\right)$ ■

KECERUNAN DUA GARIS LURUS SELARI

Dalam rajah berikut, L_1 dan L_2 merupakan garis lurus selari. Tunjukkan bahawa kecerunan kedua-dua garis itu, m_1 dan m_2, adalah sama.

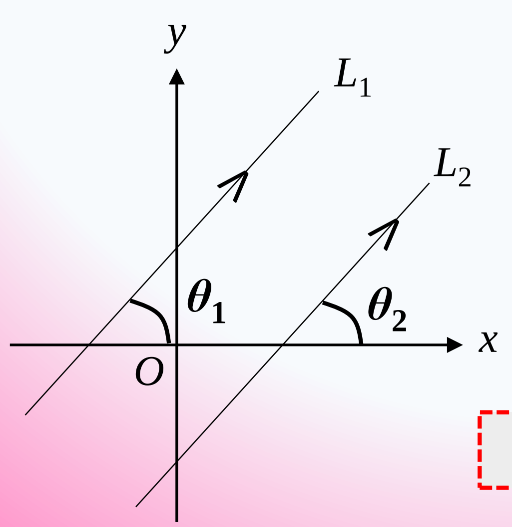
In the following diagram, L_1 and L_2 are two straight parallel lines. Show that the gradients of both lines, m_1 and m_2, are the same.



Penyelesaian / *Solution*:

Oleh sebab L_1 dan L_2 adalah dua garis lurus, sudut yang dibentuk oleh kedua-dua garis lurus dengan arah positif paksi-x ialah θ_1 dan θ_2 masing-masing.

Because L_1 and L_2 are two straight lines, the angles formed by both straight lines with the positive direction of x-axis are θ_1 and θ_2 respectively.



$$\theta_1 = \theta_2$$
$$\tan\theta_1 = \tan\theta_2$$
$$m_1 = m_2 \quad \blacksquare$$

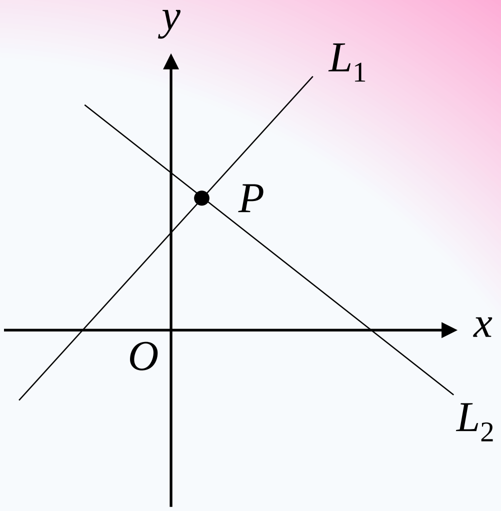
$\tan\theta = m$

Sudut sepadan
Corresponding angles

KECERUNAN DUA GARIS LURUS SERENJANG

Dalam rajah berikut, L_1 dan L_2 merupakan dua garis lurus dengan kecerunan m_1 dan m_2 masing-masing. Jika L_1 dan L_2 berserenjang di titik P, tunjukkan

In the following diagram, L_1 and L_2 are two straight lines with gradients m_1 and m_2 respectively. If L_1 and L_2 are perpendicular at point P, show that

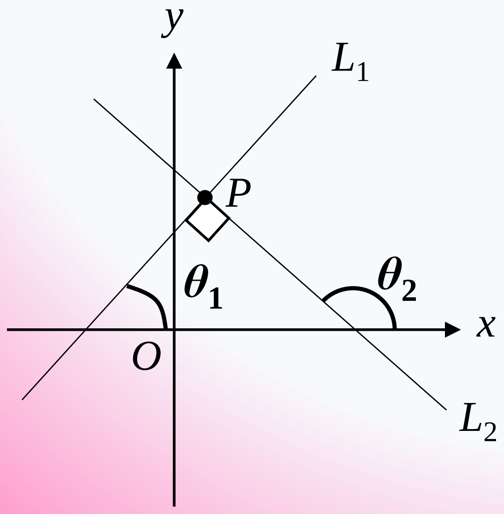


$$m_1 m_2 = -1$$

Penyelesaian / *Solution*:

θ_1 dan θ_2 ialah sudut yang dibentuk oleh L_1 dan L_2 dengan arah positif paksi-x masing-masing.

θ_1 and θ_2 are the angles formed by L_1 and L_2 with the positive direction of x-axis respectively.



$$\theta_1 + 90^\circ = \theta_2$$

$$\tan(\theta_1 + 90^\circ) = \tan\theta_2$$

$$-\frac{1}{\tan\theta_1} = \tan\theta_2$$

$$-1 = \tan\theta_2 \tan\theta_1$$

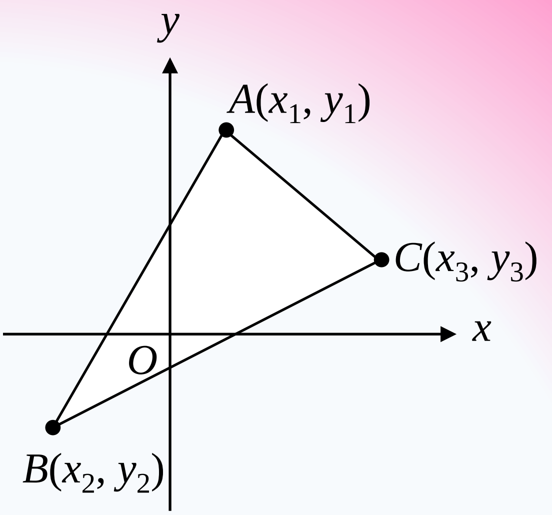
$$m_1 m_2 = -1 \qquad \blacksquare$$

Rujuk m/s 56
Refer to page 56

MENERBITKAN LUAS SEGI TIGA MENGGUNAKAN BUCU-BUCU KOORDINAT CARTES

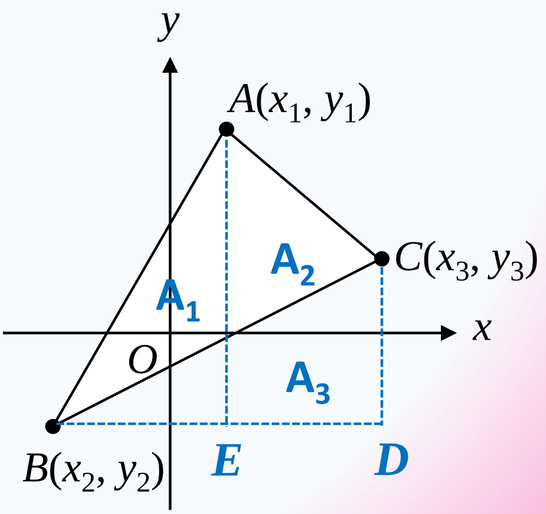
Dalam rajah berikut, A, B dan C merupakan bucu-bucu bagi sebuah segi tiga di atas satu satah Cartes. Buktikan bahawa rumus luas segi tiga itu ialah

In the following diagram, A, B and C are the vertices of a triangle on a Cartesian plane. Prove that the formula of the area of triangle is



$$\frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3)$$

Penyelesaian / *Solution*:



A_1 : segi tiga ABE

$$A_1 = \frac{1}{2}(x_1 - x_2)(y_1 - y_2)$$

$$= \frac{1}{2}(x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2)$$

A_2 : trapezium $ACDE$

$$A_2 = \frac{1}{2}\left[(y_1 - y_2) + (y_3 - y_2)\right](x_3 - x_1)$$

$$= \frac{1}{2}(x_3y_1 - x_1y_1 - 2x_3y_2 + 2x_1y_2 + x_3y_3 - x_1y_3)$$

Bersambung

A_3 : segi tiga BCD

$$A_3 = \frac{1}{2}(x_3 - x_2)(y_3 - y_2)$$

$$= \frac{1}{2}(x_3y_3 - x_3y_2 - x_2y_3 + x_2y_2)$$

Luas segi tiga $= A_1 + A_2 - A_3$

$$= \frac{1}{2}(x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2) +$$

$$\frac{1}{2}(x_3y_1 - x_1y_1 - 2x_3y_2 + 2x_1y_2 + x_3y_3 - x_1y_3) -$$

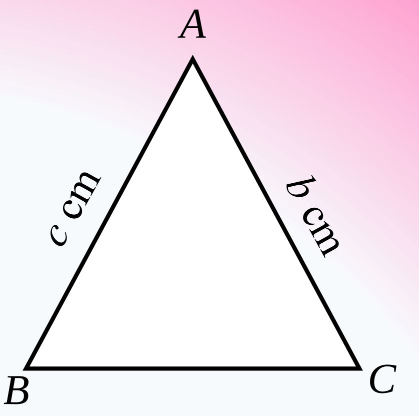
$$\frac{1}{2}(x_3y_3 - x_3y_2 - x_2y_3 + x_2y_2)$$

$$= \frac{1}{2}\begin{pmatrix} \cancel{x_1y_1} - \cancel{x_1y_2} - x_2y_1 + \cancel{x_2y_2} + x_3y_1 - \cancel{x_1y_1} - \cancel{2x_3y_2} + \\ \cancel{2}x_1y_2 + \cancel{x_3y_3} - x_1y_3 - \cancel{x_3y_3} + \cancel{x_3y_2} + x_2y_3 - \cancel{x_2y_2} \end{pmatrix}$$

$$= \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3) \qquad \blacksquare$$

MENERBITKAN PETUA SINUS

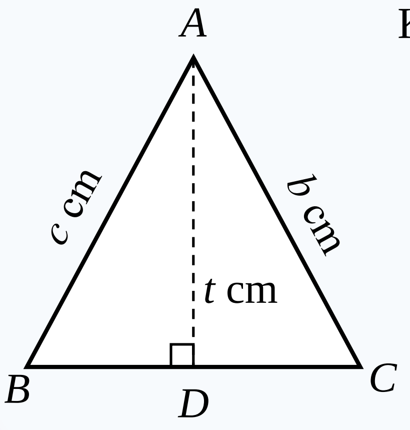
Rajah berikut menunjukkan segi tiga ABC. Panjang sisi AB dan AC ialah c cm dan b cm masing-masing.
The following diagram shows a triangle ABC. The length of sides AB and AC are c cm *and* b cm *respectively*.



Buktikan bahawa
Prove that

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

Penyelesaian / *Solution*:



Katakan tinggi segi tiga ialah $AD = t$ cm

$$\sin B = \frac{t}{c}$$

$$t = c \sin B \qquad \ldots ①$$

$$\sin C = \frac{t}{b}$$

$$t = b \sin C \qquad \ldots ②$$

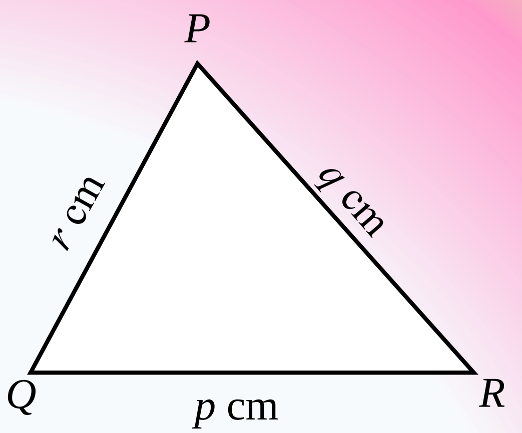
② = ①:

$$b \sin C = c \sin B$$

$$\frac{b}{\sin B} = \frac{c}{\sin C} \qquad \blacksquare$$

MENTAHKIKKAN PETUA KOSINUS

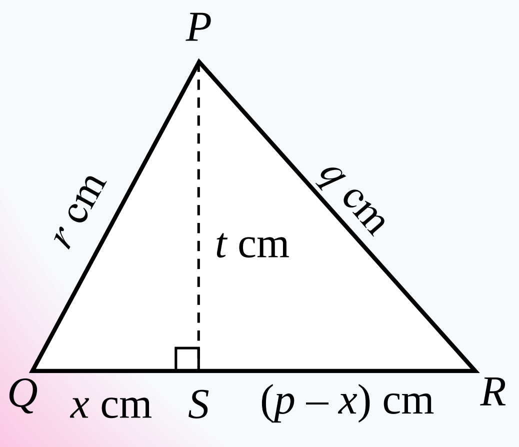
Rajah berikut menunjukkan segi tiga PQR. Panjang sisi PQ, PR dan QR ialah r cm, q cm dan p cm masing-masing.
The following diagram shows a triangle PQR. The length of sides PQ, PR and QR are r cm, q cm and p cm respectively.



Tahkikkan bahawa $q^2 = p^2 + r^2 - 2pr \text{ kos } Q$.
Verify that $q^2 = p^2 + r^2 - 2pr \text{ kos } Q$.

Penyelesaian / *Solution*:

Katakan tinggi segi tiga ialah $PS = t$ cm
Biarkan $QS = x$ cm dan $SR = (p - x)$ cm



$$q^2 = t^2 + (p - x)^2$$
$$q^2 = t^2 + p^2 - 2px + x^2 \qquad \ldots①$$

$$t^2 = r^2 - x^2 \qquad \ldots②$$

$$\text{kos } Q = \frac{x}{r}$$
$$x = r \text{ kos } Q \qquad \ldots③$$

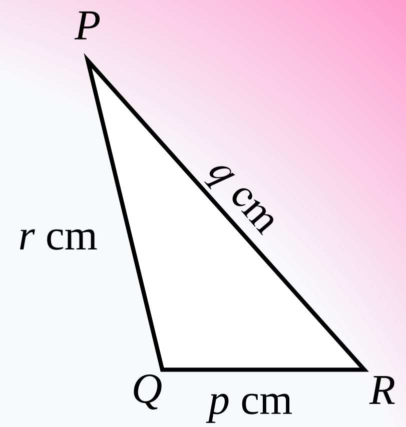
Teorem Pythagoras
Pythagoras' Theorem

Ganti ③ dan ② dalam ①:

$$q^2 = (r^2 - x^2) + p^2 - 2p(r \text{ kos } Q) + x^2$$
$$q^2 = r^2 + p^2 - 2pr \text{ kos } Q \qquad \blacksquare$$

MENTAHKIKKAN PETUA KOSINUS

Rajah berikut menunjukkan segi tiga PQR. Panjang sisi PQ, PR dan QR ialah r cm, q cm dan p cm masing-masing.
The following diagram shows a triangle PQR. The length of sides PQ, PR and QR are r cm, q cm and p cm respectively.



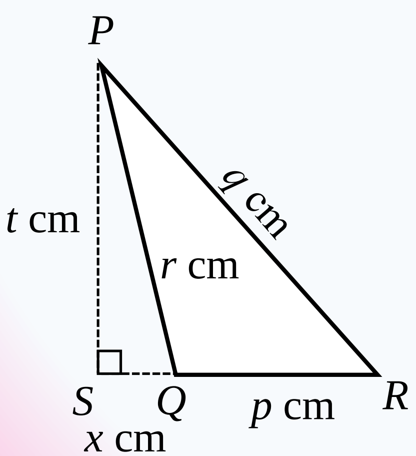
Tahkikkan bahawa $q^2 = p^2 + r^2 - 2pr \text{ kos } Q$.
Verify that $q^2 = p^2 + r^2 - 2pr \text{ kos } Q$.

Penyelesaian / *Solution*:

Panjangkan sisi agar membentuk segi tiga bersudut tegak

Katakan tinggi segi tiga ialah $PS = t$ cm

Biarkan $SQ = x$ cm dan $SR = (p + x)$ cm



$$q^2 = t^2 + (p+x)^2$$

$$q^2 = t^2 + p^2 + 2px + x^2 \qquad \ldots①$$

$$t^2 = r^2 - x^2 \qquad \ldots②$$

$$\text{kos } Q = -\frac{x}{r}$$

$$x = -r \text{ kos } Q \qquad \ldots③$$

Teorem Pythagoras
Pythagoras' Theorem

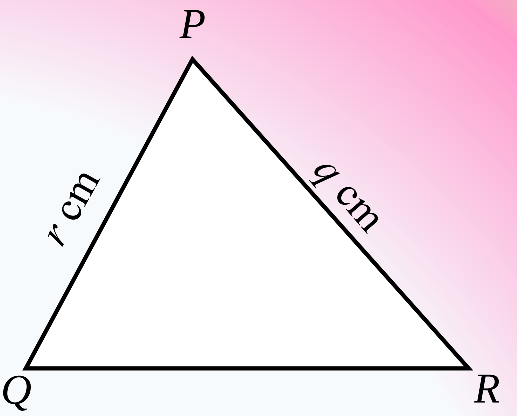
Ganti ③ dan ② dalam ①:

$$q^2 = (r^2 - x^2) + p^2 + 2p(-r \text{ kos } Q) + x^2$$

$$q^2 = r^2 + p^2 - 2pr \text{ kos } Q \qquad \blacksquare$$

MENERBITKAN RUMUS LUAS SEGI TIGA

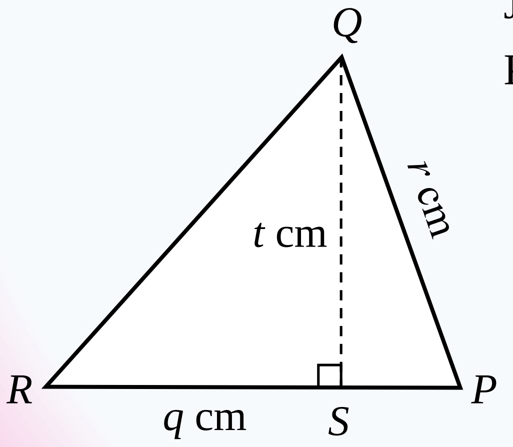
Rajah berikut menunjukkan segi tiga PQR. Panjang sisi PQ dan PR ialah r cm dan q cm masing-masing.
The following diagram shows a triangle PQR. The length of sides PQ and PR are r cm *and* q cm *respectively*.



Tunjukkan luas segi tiga itu ialah
Show that the area of the triangle is

$$\frac{1}{2}qr\sin P$$

Penyelesaian / *Solution*:



Jadikan PR sebagai tapak segi tiga PQR
Katakan tinggi segi tiga ialah $QS = t$ cm

$$\text{Luas } PQR = \frac{1}{2}qt \qquad \ldots①$$

$$\sin P = \frac{t}{r}$$

$$t = r\sin P \qquad \ldots②$$

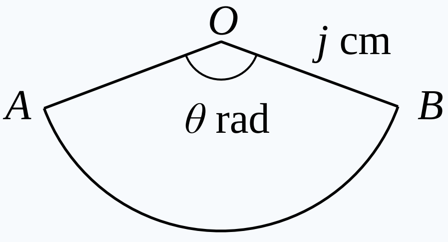
Ganti ② dalam ①:

$$\text{Luas } PQR = \frac{1}{2}qr\sin P \qquad \blacksquare$$

TINGKATAN 5

RUMUS PANJANG LENGKOK SUATU BULATAN

Rajah menunjukkan sebuah sektor *OAB* dengan pusat *O*.
Diagram shows the sector OAB with centre O.



Tunjukkan bahawa panjang lengkok *AB*, *s* diungkapkan sebagai
Show that the length of arc AB, s is expressed as

$$s = j\theta$$

Penyelesaian / *Solution*:

$$\frac{\text{panjang lengkok}}{\text{lilitan bulatan}} = \frac{\text{sudut yang dicangkum di pusat}}{360^\circ}$$

$$\frac{s}{2\pi j} = \frac{\theta \text{ rad}}{2\pi \text{ rad}}$$

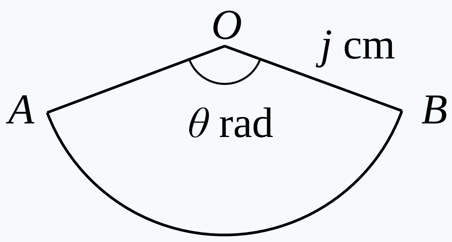
Mansuhkan 2π
Cancel off 2π

$$\frac{s}{j} = \theta$$

$$s = j\theta \qquad \blacksquare$$

RUMUS LUAS SEKTOR SUATU BULATAN

Rajah menunjukkan sebuah sektor OAB dengan pusat O.
Diagram shows the sector OAB with centre O.



Tunjukkan bahawa luas sektor OAB, L diungkapkan sebagai
Show that the area of sector OAB, L is expressed as

$$L = \frac{1}{2} j^2 \theta$$

Penyelesaian / *Solution*:

$$\frac{\text{luas sektor}}{\text{luas bulatan}} = \frac{\text{sudut yang dicangkum di pusat}}{360^\circ}$$

$$\frac{L}{\pi j^2} = \frac{\theta \text{ rad}}{2\pi \text{ rad}}$$

Mansuhkan π
Cancel off π

$$\frac{L}{j^2} = \frac{\theta}{2}$$

$$L = \frac{1}{2} j^2 \theta \qquad \blacksquare$$

TERBITAN PERTAMA SECARA INDUKTIF

Buat satu kesimpulan umum secara induktif bagi terbitan pertama fungsi yang mengikut pola berikut.
Make a general conclusion by induction for the first derivative of functions which follows the following pattern.

Fungsi *Function*	Terbitan Pertama *First Derivative*
$y = 6x$	$\frac{dy}{dx} = 6$
$y = 6x^2$	$\frac{dy}{dx} = 12x$
$y = 6x^3$	$\frac{dy}{dx} = 18x^2$
$\dots$	$\dots$
$y = 6x^n$	$\frac{dy}{dx} = \cdots$

Penyelesaian / *Solution*:

$$\text{Jika } y = 6x^n \text{ , maka } \frac{dy}{dx} = 6nx^{n-1} \qquad \blacksquare$$

TERBITAN PERTAMA SECARA INDUKTIF

Buat satu kesimpulan umum secara induktif bagi terbitan pertama fungsi yang mengikut pola berikut.
Make a general conclusion by induction for the first derivative of functions which follows the following pattern.

Fungsi *Function*	Terbitan Pertama *First Derivative*
$y = 6x$	$\frac{dy}{dx} = 6$
$y = 6x^2$	$\frac{dy}{dx} = 6(2)x^{2-1}$
$y = 6x^3$	$\frac{dy}{dx} = 6(3)x^{3-1}$
$\ldots$	$\ldots$
$y = 6x^n$	$\frac{dy}{dx} = \cdots$

Penyelesaian / *Solution*:

$$\text{Jika } y = 6x^n \text{ , maka } \frac{dy}{dx} = 6nx^{n-1} \qquad \blacksquare$$

PEMBUKTIAN TERBITAN PERTAMA HASIL TAMBAH DUA FUNGSI MENGGUNAKAN IDEA HAD

Diberi dua fungsi, $f(x)$ dan $g(x)$. Menggunakan pembezaan dengan prinsip pertama, buktikan
Given two functions, $f(x)$ and $g(x)$. Using the first principle of differentiation, prove that

$$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$$

Penyelesaian / *Solution*:

$$\frac{dy}{dx} = \underset{\delta x \to 0}{\text{had}} \frac{f(x+\delta x)-f(x)}{\delta x}$$

Katakan $y = f(x)+g(x)$

$$y+\delta y = f(x+\delta x)+g(x+\delta x)$$
$$\delta y = f(x+\delta x)+g(x+\delta x)-f(x)-g(x)$$

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x)+g(x+\delta x)-f(x)-g(x)}{\delta x}$$
$$\underset{\delta x \to 0}{\text{had}} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\text{had}} \frac{f(x+\delta x)+g(x+\delta x)-f(x)-g(x)}{\delta x}$$
$$\frac{dy}{dx} = \underset{\delta x \to 0}{\text{had}} \frac{f(x+\delta x)-f(x)}{\delta x} + \underset{\delta x \to 0}{\text{had}} \frac{g(x+\delta x)-g(x)}{\delta x}$$
$$\frac{d}{dx}\left[f(x)+g(x)\right] = f'(x)+g'(x) \qquad \blacksquare$$

PEMBUKTIAN TERBITAN PERTAMA HASIL DARAB DUA FUNGSI MENGGUNAKAN IDEA HAD

Diberi dua fungsi, $u(x)$ dan $v(x)$. Menggunakan pembezaan dengan prinsip pertama, buktikan
Given two functions, $u(x)$ and $v(x)$. Using the first principle of differentiation, prove that

$$\frac{d}{dx}[u(x)v(x)] = u(x)v'(x) + v(x)u'(x)$$

Penyelesaian / *Solution*:

Katakan $y = u(x)v(x)$

$$y + \delta y = u(x+\delta x)v(x+\delta x)$$
$$\delta y = u(x+\delta x)v(x+\delta x) - u(x)v(x)$$
$$\delta y = u(x+\delta x)v(x+\delta x) - u(x+\delta x)v(x) + u(x+\delta x)v(x) - u(x)v(x)$$
$$\delta y = u(x+\delta x)\left[v(x+\delta x) - v(x)\right] + v(x)\left[u(x+\delta x) - u(x)\right]$$

Bersambung

$$\frac{\delta y}{\delta x}=\frac{u(x+\delta x)\left[v(x+\delta x)-v(x)\right]+v(x)\left[u(x+\delta x)-u(x)\right]}{\delta x}$$

$$\operatorname*{had}_{\delta x\to 0}\frac{\delta y}{\delta x}=\operatorname*{had}_{\delta x\to 0}\frac{u(x+\delta x)\left[v(x+\delta x)-v(x)\right]+v(x)\left[u(x+\delta x)-u(x)\right]}{\delta x}$$

$$\frac{dy}{dx}=\operatorname*{had}_{\delta x\to 0}\frac{u(x+\delta x)\left[v(x+\delta x)-v(x)\right]}{\delta x}+\operatorname*{had}_{\delta x\to 0}\frac{v(x)\left[u(x+\delta x)-u(x)\right]}{\delta x}$$

$$\frac{dy}{dx}=\operatorname*{had}_{\delta x\to 0}u(x+\delta x)\operatorname*{had}_{\delta x\to 0}\frac{\left[v(x+\delta x)-v(x)\right]}{\delta x}+\operatorname*{had}_{\delta x\to 0}v(x)\operatorname*{had}_{\delta x\to 0}\frac{\left[u(x+\delta x)-u(x)\right]}{\delta x}$$

$$\frac{dy}{dx}=u(x)\operatorname*{had}_{\delta x\to 0}\frac{v(x+\delta x)-v(x)}{\delta x}+v(x)\operatorname*{had}_{\delta x\to 0}\frac{u(x+\delta x)-u(x)}{\delta x}$$

$$\frac{d}{dx}\left[u(x)v(x)\right]=u(x)v'(x)+v(x)u'(x) \qquad \blacksquare$$

PEMBUKTIAN TERBITAN PERTAMA HASIL DARAB DUA FUNGSI MENGGUNAKAN IDEA HAD

Diberi dua fungsi, $u(x)$ dan $v(x)$. Menggunakan pembezaan dengan prinsip pertama, buktikan
Given two functions, $u(x)$ and $v(x)$. Using the first principle of differentiation, prove that

$$\frac{d}{dx}[u(x)v(x)] = u(x)v'(x) + v(x)u'(x)$$

Penyelesaian / *Solution*:

KAEDAH ALTERNATIF
ALTERNATIVE METHOD

Katakan $y = u(x)v(x) = uv$

$$y + \delta y = (u + \delta u)(v + \delta v)$$
$$\delta y = (u + \delta u)(v + \delta v) - uv$$
$$\delta y = uv + u\delta v + v\delta u + \delta u\delta v - uv$$
$$\delta y = u\delta v + v\delta u + \delta u\delta v$$

$$\frac{\delta y}{\delta x} = \frac{u\delta v + v\delta u + \delta u\delta v}{\delta x}$$
$$\underset{\delta x \to 0}{\text{had}} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\text{had}} \frac{u\delta v + v\delta u + \delta u\delta v}{\delta x}$$

$$\frac{dy}{dx} = \underset{\delta x \to 0}{\text{had}} \frac{\delta y}{\delta x} = y'$$

Apabila $\delta x \to 0, \delta y \to 0, \delta u \to 0$ dan $\delta v \to 0$

$$\frac{dy}{dx} = \underset{\delta x \to 0}{\text{had}} \frac{u\delta v}{\delta x} + \underset{\delta x \to 0}{\text{had}} \frac{v\delta u}{\delta x} + \underset{\delta v \to 0}{\text{had}} \frac{\delta u}{\delta x}\delta v$$
$$\frac{dy}{dx} = \underset{\delta x \to 0}{\text{had}}\, u \underset{\delta x \to 0}{\text{had}} \frac{\delta v}{\delta x} + \underset{\delta x \to 0}{\text{had}}\, v \underset{\delta x \to 0}{\text{had}} \frac{\delta u}{\delta x} + 0$$

$$\frac{d}{dx}\left[u(x)v(x)\right] = u(x)v'(x) + v(x)u'(x) \qquad \blacksquare$$

PEMBUKTIAN TERBITAN PERTAMA HASIL BAHAGI DUA FUNGSI MENGGUNAKAN IDEA HAD

Diberi dua fungsi, $u(x)$ dan $v(x)$, $v(x) \neq 0$. Menggunakan pembezaan dengan prinsip pertama, buktikan
Given two functions, $u(x)$ and $v(x)$, $v(x) \neq 0$. Using the first principle of differentiation, prove that

$$\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2}$$

Penyelesaian / *Solution*:

Katakan $y = \dfrac{u(x)}{v(x)}$

$$y + \delta y = \frac{u(x+\delta x)}{v(x+\delta x)}$$

$$\delta y = \frac{u(x+\delta x)}{v(x+\delta x)} - \frac{u(x)}{v(x)}$$

$$\delta y = \frac{v(x)u(x+\delta x) - u(x)v(x+\delta x)}{v(x)v(x+\delta x)}$$

$$\delta y = \frac{v(x)u(x+\delta x) - u(x)v(x) + u(x)v(x) - u(x)v(x+\delta x)}{v(x)v(x+\delta x)}$$

$$\delta y = \frac{v(x)\left[u(x+\delta x) - u(x)\right] - u(x)\left[v(x+\delta x) - v(x)\right]}{v(x)v(x+\delta x)}$$

Bersambung

$$\frac{\delta y}{\delta x}=\frac{1}{\delta x}\left(\frac{v(x)\left[u(x+\delta x)-u(x)\right]-u(x)\left[v(x+\delta x)-v(x)\right]}{v(x)v(x+\delta x)}\right)$$

$$\underset{\delta x\to 0}{\text{had}}\frac{\delta y}{\delta x}=\underset{\delta x\to 0}{\text{had}}\frac{1}{v(x)v(x+\delta x)}\left(\frac{v(x)\left[u(x+\delta x)-u(x)\right]-u(x)\left[v(x+\delta x)-v(x)\right]}{\delta x}\right)$$

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{had}}\frac{1}{v(x)v(x+\delta x)}\left(\begin{array}{l}\underset{\delta x\to 0}{\text{had}}\dfrac{v(x)\left[u(x+\delta x)-u(x)\right]}{\delta x}-\\ \qquad\qquad\underset{\delta x\to 0}{\text{had}}\dfrac{u(x)\left[v(x+\delta x)-v(x)\right]}{\delta x}\end{array}\right)$$

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{had}}\frac{1}{v(x)v(x+\delta x)}\left(\begin{array}{l}\underset{\delta x\to 0}{\text{had}}\,v(x)\underset{\delta x\to 0}{\text{had}}\dfrac{u(x+\delta x)-u(x)}{\delta x}-\\ \qquad\qquad\underset{\delta x\to 0}{\text{had}}\,u(x)\underset{\delta x\to 0}{\text{had}}\dfrac{v(x+\delta x)-v(x)}{\delta x}\end{array}\right)$$

$$\frac{dy}{dx}=\frac{1}{v(x)v(x)}\left[v(x)u'(x)-u(x)v'(x)\right]$$

$$\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right]=\frac{v(x)u'(x)-u(x)v'(x)}{\left[v(x)\right]^2}\qquad\blacksquare$$

PEMBUKTIAN TERBITAN PERTAMA HASIL BAHAGI DUA FUNGSI MENGGUNAKAN IDEA HAD

Diberi dua fungsi, $u(x)$ dan $v(x)$, $v(x) \neq 0$. Menggunakan pembezaan dengan prinsip pertama, buktikan
Given two functions, $u(x)$ and $v(x)$, $v(x) \neq 0$. Using the first principle of differentiation, prove that

$$\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2}$$

Penyelesaian / *Solution*:

KAEDAH ALTERNATIF
ALTERNATIVE METHOD

Katakan $y = \frac{u(x)}{v(x)} = \frac{u}{v}$

$$y + \delta y = \frac{u + \delta u}{v + \delta v}$$

$$\delta y = \frac{u + \delta u}{v + \delta v} - \frac{u}{v}$$

$$\delta y = \frac{uv + v\delta u - uv - u\delta v}{v(v + \delta v)}$$

$$\delta y = \frac{v\delta u - u\delta v}{v(v + \delta v)}$$

$$\frac{\delta y}{\delta x} = \frac{\frac{v\delta u}{\delta x} - \frac{u\delta v}{\delta x}}{v(v + \delta v)}$$

$$\operatorname*{had}_{\delta x \to 0} \frac{\delta y}{\delta x} = \operatorname*{had}_{\delta x \to 0} \frac{1}{v(v + \delta v)}\left[\frac{v\delta u}{\delta x} - \frac{u\delta v}{\delta x}\right]$$

Apabila $\delta x \to 0, \delta y \to 0, \delta u \to 0$ dan $\delta v \to 0$

$$\frac{dy}{dx} = \underset{\delta v \to 0}{\text{had}} \frac{1}{v(v+\delta v)} \underset{\delta x \to 0}{\text{had}} \left[\frac{v\delta u}{\delta x} - \frac{u\delta v}{\delta x} \right]$$

$$\frac{dy}{dx} = \underset{\delta x \to 0}{\text{had}} \frac{\delta y}{\delta x} = y'$$

$$\frac{dy}{dx} = \frac{1}{v(v+0)} \left[v \underset{\delta x \to 0}{\text{had}} \frac{\delta u}{\delta x} - u \underset{\delta x \to 0}{\text{had}} \frac{\delta v}{\delta x} \right]$$

$$\frac{dy}{dx} = \frac{1}{v^2}(vu' - uv')$$

$$\frac{d}{dx}\left[\frac{u(x)}{v(x)} \right] = \frac{v(x)u'(x) - u(x)v'(x)}{\left[v(x) \right]^2} \quad \blacksquare$$

PEMBUKTIAN PETUA RANTAI MENGGUNAKAN IDEA HAD

Diberi dua fungsi, $y = f(u)$ dan $u = g(x)$. Menggunakan idea had, buktikan
Given two functions, $y = f(u)$ and $u = g(x)$. Using the idea of limits, prove that

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Penyelesaian / *Solution*:

$$\frac{dy}{dx} = \underset{\delta x \to 0}{\text{had}} \frac{\delta y}{\delta x}$$

$$\frac{dy}{dx} = \underset{\delta x \to 0}{\text{had}} \left(\frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x} \right)$$

$$\frac{dy}{dx} = \underset{\delta x \to 0}{\text{had}} \frac{\delta y}{\delta u} \times \underset{\delta x \to 0}{\text{had}} \frac{\delta u}{\delta x}$$

$$\boxed{\frac{dy}{dx} = \underset{\delta x \to 0}{\text{had}} \frac{\delta y}{\delta x}}$$

Apabila $\delta x \to 0, \delta y \to 0$ dan $\delta u \to 0$

$$\frac{dy}{dx} = \underset{\delta u \to 0}{\text{had}} \frac{\delta y}{\delta u} \times \underset{\delta x \to 0}{\text{had}} \frac{\delta u}{\delta x}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \qquad \blacksquare$$

MENERBITKAN RUMUS KAMIRAN TAK TENTU SECARA INDUKTIF

Buat satu kesimpulan umum secara induktif bagi kamiran tak tentu suatu fungsi yang mengikut pola berikut.
Make a general conclusion by induction for the indefinite integral of functions which follows the following pattern.

Fungsi *Function*	Kamiran tak tentu *Indefinite integral*
$y = 6x$	$\int 6x\,dx = 3x^2 + c$
$y = 6x^2$	$\int 6x^2\,dx = 2x^3 + c$
$y = 6x^3$	$\int 6x^3\,dx = \frac{3}{2}x^4 + c$
$\ldots$	$\ldots$
$y = 6x^n$	$\int 6x^n\,dx = \cdots$

Tuliskan satu syarat bagi nilai n dan namakan c.
Write one condition for the value of n and name c.

Penyelesaian / *Solution*:

$$\text{Jika } y = 6x^n \text{, maka } \int 6x^n\,dx = \frac{6}{n+1}x^{n+1} + c \quad \blacksquare$$

$n \neq -1$, c ialah suatu pemalar sembarangan.

$n \neq -1$, *c is an arbitrary constant.* $\blacksquare$

MENERBITKAN RUMUS KAMIRAN TAK TENTU SECARA INDUKTIF

Buat satu kesimpulan umum secara induktif bagi kamiran tak tentu suatu fungsi yang mengikut pola berikut.
Make a general conclusion by induction for the indefinite integral of functions which follows the following pattern.

Fungsi *Function*	Kamiran tak tentu *Indefinite integral*
$y = 6x$	$\int 6x\,dx = \frac{6}{1+1}x^{1+1} + c$
$y = 6x^2$	$\int 6x^2\,dx = \frac{6}{2+1}x^{2+1} + c$
$y = 6x^3$	$\int 6x^3\,dx = \frac{6}{3+1}x^{3+1} + c$
$\dots$	$\dots$
$y = 6x^n$	$\int 6x^n\,dx = \cdots$

Tuliskan satu syarat bagi nilai n dan namakan c.
Write one condition for the value of n and name c.

Penyelesaian / *Solution*:

$$\text{Jika } y = 6x^n \text{, maka } \int 6x^n\,dx = \frac{6}{n+1}x^{n+1} + c \qquad \blacksquare$$

$n \neq -1$, c ialah suatu pemalar sembarangan.

$n \neq -1$, c *is an arbitrary constant.* $\blacksquare$

RUMUS KAMIRAN FUNGSI $(ax+b)^n$, $n \neq -1$

Tunjukkan
Show that

$$\int (ax+b)^n \ dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

dengan a, b dan c ialah pemalar dan n ialah sebarang nombor nyata, $n \neq -1$.
where a, b and c are constants and n is any real number, $n \neq -1$.

Penyelesaian / *Solution*:

Katakan $u = ax + b$

$$\frac{du}{dx} = a$$

$$dx = \frac{du}{a}$$

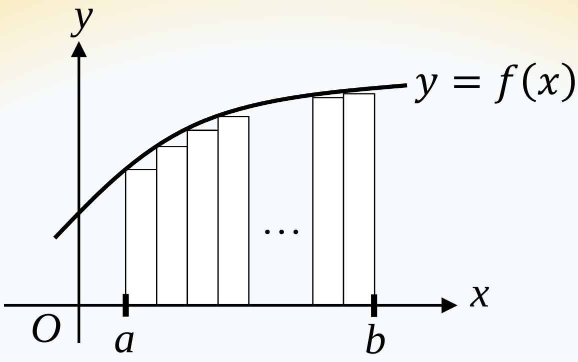
$$\int (ax+b)^n \, dx = \int u^n \frac{du}{a}$$

$$= \frac{1}{a}\int u^n \, du$$

$$= \frac{1}{a}\left(\frac{u^{n+1}}{n+1}\right) + c \qquad , n \neq -1$$

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c \qquad , n \neq -1 \qquad \blacksquare$$

MENERBITKAN RUMUS AM LUAS DI BAWAH LENGKUNG



Rajah menunjukkan lengkung $y = f(x)$. Luas bawah lengkung itu dari $x = a$ hingga $x = b$ diwakili oleh n jalur segi empat tepat yang nipis dengan lebar yang seragam, dengan keadaan n ialah suatu integer positif.
Diagram shows a curve $y = f(x)$. The area under the curve from $x = a$ to $x = b$ is represented by n strips of thin rectangle with uniform width, where n is a positive integer.

Terbitkan rumus luas di bawah lengkung itu apabila $n \to \infty$.
Derive the formula of the area under the curve as $n \to \infty$.

Penyelesaian / *Solution*:

Lebar seragam jalur segi empat tepat, $\delta x = \dfrac{b-a}{n}$

Setiap jalur segi empat tepat mempunyai ketinggian y_i

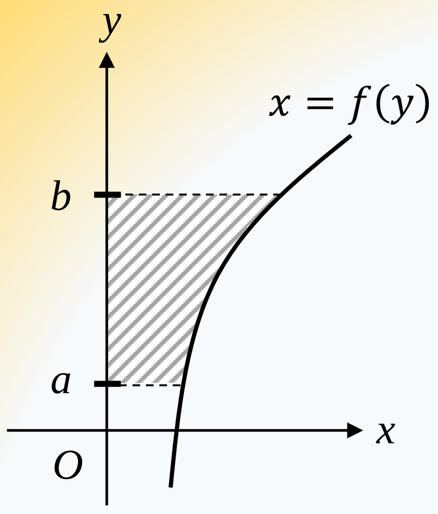
Luas satu jalur segi empat tepat tertentu ialah $y_i \delta x$

Jumlah luas n jalur segi empat tepat, $L = \sum_{i=1}^{n} y_i \delta x$

Apabila $n \to \infty, \delta x \to 0$:
$$L = \underset{\delta x \to 0}{\text{had}} \sum_{i=1}^{n} y_i \delta x$$

$$L = \int_a^b y \, dx \qquad \blacksquare$$

MENERBITKAN RUMUS AM LUAS DI BAWAH LENGKUNG



Rajah menunjukkan lengkung $x = f(y)$.
Diagram shows a curve $x = f(y)$.

(*a*) Nyatakan rumus am luas rantau berlorek dari $y = a$ hingga $y = b$.
State the general formula of the area of the shaded region from $y = a$ *to* $y = b$.

(*b*) Tunjukkan bagaimanakah rumus di (*a*) diterbitkan.
Show how the formula in (*a*) *is derived*.

Penyelesaian / *Solution*:

(*a*) $\int_a^b x\,dy$

(*b*) Luas yang terhasil ialah penghasiltambahan n jalur segi empat tepat

Lebar seragam jalur segi empat tepat, $\delta y = \dfrac{b-a}{n}$

Setiap jalur segi empat tepat mempunyai panjang x_i

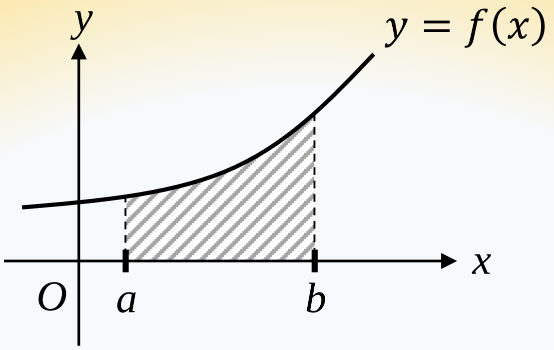
Luas satu jalur segi empat tepat tertentu ialah $x_i \delta y$

Jumlah luas n jalur segi empat tepat, $L = \sum_{i=1}^{n} x_i \delta y$

Apabila $n \to \infty, \delta y \to 0$: $\quad L = \operatorname*{had}_{\delta y \to 0} \sum_{i=1}^{n} x_i \delta y$

$$L = \int_a^b x\,dy \qquad \blacksquare$$

MENERBITKAN RUMUS AM ISIPADU KISARAN



Rajah menunjukkan lengkung $y = f(x)$.
Diagram shows a curve $y = f(x)$.

(*a*) Nyatakan rumus am isi padu kisaran, dalam sebutan π apabila rantau berlorek diputarkan melalui 360° pada paksi-*x* dari $x = a$ hingga $x = b$.
State the general formula of the volume of revolution, in terms of π, *when the shaded region is rotated* 360° *about the x-axis from* $x = a$ *to* $x = b$.

(*b*) Tunjukkan bagaimanakah rumus di (*a*) diterbitkan.
Show how the formula in (*a*) *is derived*.

Penyelesaian / *Solution*:

(*a*) $\pi\int_a^b y^2\ dx$

(*b*) Isipadu yang terhasil ialah penghasiltambahan *n* silinder diskret

Ketebalan silinder, $\delta x = \dfrac{b-a}{n}$; Jejari silinder y_i

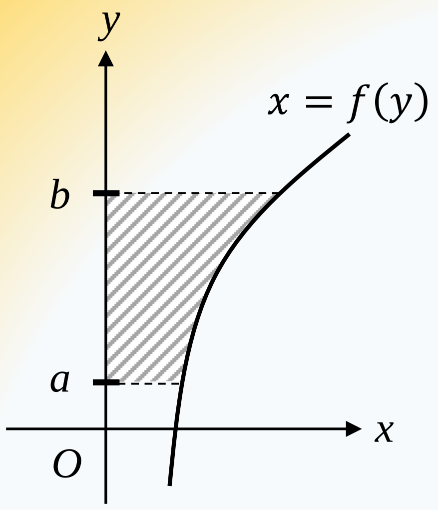
Isipadu satu silinder diskret ialah $\pi y_i^2 \delta x$

Jumlah isipadu *n* silinder diskret, $I = \sum_{i=1}^{n} \pi y_i^2 \delta x$

Apabila $n \to \infty, \delta x \to 0$: $\quad I = \underset{\delta x \to 0}{\text{had}} \sum_{i=1}^{n} \pi y_i^2 \delta x$

$$I = \pi\int_a^b y^2\ dx \qquad \blacksquare$$

MENERBITKAN RUMUS AM ISIPADU KISARAN



Rajah menunjukkan lengkung $x = f(y)$.
Diagram shows a curve $x = f(y)$.

(*a*) Nyatakan rumus am isi padu kisaran, dalam sebutan π apabila rantau berlorek diputarkan melalui 360° pada paksi-y dari $y = a$ hingga $y = b$.
State the general formula of the volume of revolution, in terms of π, when the shaded region is rotated 360° about the y-axis from $y = a$ to $y = b$.

(*b*) Tunjukkan bagaimanakah rumus di (*a*) diterbitkan.
Show how the formula in (a) is derived.

Penyelesaian / *Solution*:

(*a*) $\pi\int_a^b x^2\,dy$

(*b*) Isipadu yang terhasil ialah penghasiltambahan n silinder diskret

Ketebalan silinder, $\delta y = \dfrac{b-a}{n}$; Jejari silinder x_i

Isipadu satu silinder diskret ialah $\pi x_i^2 \delta y$

Jumlah isipadu n silinder diskret, $I = \sum_{i=1}^{n} \pi x_i^2 \delta y$

Apabila $n \to \infty, \delta y \to 0$: $\quad I = \text{had}_{\delta y \to 0} \sum_{i=1}^{n} \pi x_i^2 \delta y$

$$I = \pi\int_a^b x^2\,dy \quad \blacksquare$$

HUBUNGAN ANTARA GABUNGAN DENGAN PILIH ATUR

Buktikan
Prove that

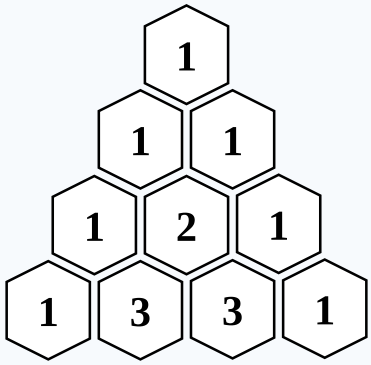
$$^{n}P_{r} = (^{n}C_{r})(r!)$$

Penyelesaian / *Solution*:

$$\begin{aligned}
\text{Sebelah kanan} &= \left(^{n}C_{r}\right)\left(r!\right) \\
&= \frac{n!}{\left(n-r\right)!r!} \times r! \\
&= \frac{n!}{\left(n-r\right)!} \\
&= {}^{n}P_{r} \\
&= \text{ Sebelah kiri} \qquad \blacksquare
\end{aligned}$$

KAITAN ^nC_r DENGAN SEGI TIGA PASCAL

Lihat segi tiga Pascal berikut.
Observe the following Pascal's triangle.



Berdasarkan pengetahuan anda tentang gabungan iaitu ^nC_r, apakah yang boleh dirumuskan daripada baris akhir segi tiga Pascal itu?
Based on your knowledge about combination which is ^nC_r, what can be concluded from the last row of the Pascal's triangle?

Seterusnya, senaraikan semua nombor dalam baris keenam segi tiga Pascal.
Hence, list all the numbers in the sixth row in the Pascal's triangle.

Penyelesaian / *Solution*:

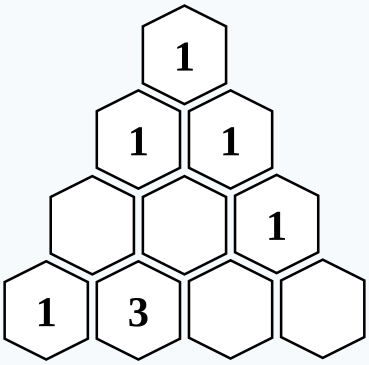
$^3C_0 = 1 \quad ; \quad ^3C_1 = 3 \quad ; \quad ^3C_2 = 3 \quad ; \quad ^3C_3 = 1$ ■

$^6C_0 = 1 \quad ; \quad ^6C_1 = 6 \quad ; \quad ^6C_2 = 15 \quad ; \quad ^6C_3 = 20$
$^6C_4 = 15 \quad ; \quad ^6C_5 = 6 \quad ; \quad ^6C_6 = 1$

$1, 6, 15, 20$ ■

MEMBUKTIKAN $\sum_{r=0}^{n} P(X = r) = 1$ BAGI SUATU TABURAN BINOMIAL

(*a*)(i) Lengkapkan segi tiga Pascal berikut.
Complete the following Pascal's triangle.



Berdasarkan pengetahuan anda tentang gabungan iaitu ${}^nC_r$, apakah yang boleh diperhatikan daripada baris akhir segi tiga Pascal itu?
Based on your knowledge about combination which is ${}^nC_r$, *what can be observed from the last row of the Pascal's triangle*?

(ii) Kembangkan $(q+p)^3$ dalam sebutan menaik p.
Expand $(q+p)^3$ *in the ascending order of* p.

(*b*) Bandingkan jawapan di (*a*)(i) dan (*a*)(ii). Sekiranya $X \sim B(3, p)$, dengan keadaan X ialah suatu pemboleh ubah rawak, tunjukkan
Compare the answers in (*a*)(i) *and* (*a*)(ii). *If* $X \sim B(3, p)$, *where* X *is a random variable, show that*

$$\sum_{r=0}^{3} P(X = r) = 1$$

Bersambung

Penyelesaian / *Solution*:

(a)(i)

1

1 1

1 2 1

1 3 3 1

$$ {}^3C_0 = 1 \quad ; \quad {}^3C_1 = 3 \quad ; \quad {}^3C_2 = 3 \quad ; \quad {}^3C_3 = 1 \qquad \blacksquare $$

(ii)

$$ \begin{aligned} (q+p)^3 &= (q+p)(q+p)^2 \\ &= (q+p)\left(q^2 + 2pq + p^2\right) \\ &= q^3 + 2pq^2 + p^2q + pq^2 + 2p^2q + p^3 \\ (q+p)^3 &= q^3 + 3pq^2 + 3p^2q + p^3 \qquad \blacksquare \end{aligned} $$

(b) $q^3 + 3pq^2 + 3p^2q + p^3 = {}^3C_0 p^0 q^{3-0} + {}^3C_1 p^1 q^{3-1} + {}^3C_2 p^2 q^{3-2} + {}^3C_3 p^3 q^{3-3}$

Diberi $X \sim B(n, p)$

Oleh itu $p + q = 1 \quad ; \quad n = 3 \quad ; \quad r = 0, 1, 2, 3$

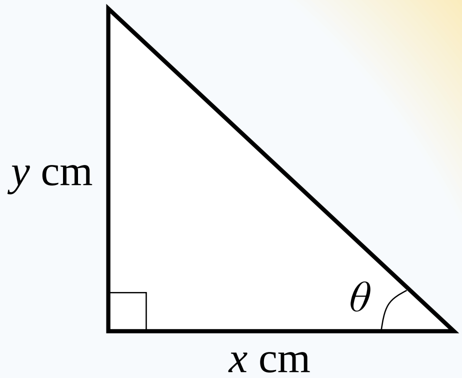
$$ \begin{aligned} (q+p)^3 &= P(X=0) + P(X=1) + P(X=2) + P(X=3) \\ 1^3 &= \sum_{r=0}^{3} {}^3C_r p^r q^{3-r} \\ 1 &= \sum_{r=0}^{3} P(X=r) \end{aligned} $$

$$ \sum_{r=0}^{3} P(X=r) = 1 \qquad \blacksquare $$

RUMUS SUDUT PELENGKAP

Rajah menunjukkan satu segi tiga bersudut tegak dengan panjang tapak x cm and tinggi y cm.
Diagram shows a right-angled triangle with a base length of x cm *and height* y cm.

Dengan menggunakan rajah itu, buktikan
By using the diagram, show that

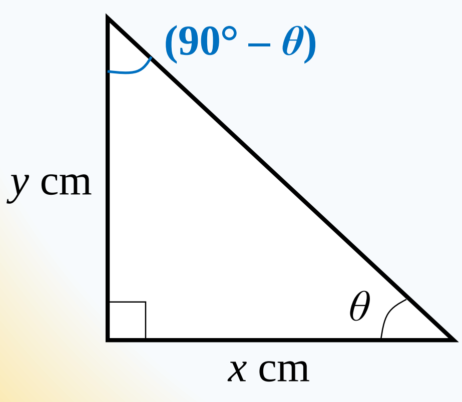


(*a*) sin θ = kos (90° – θ)
sin θ = cos (90° – θ)

(*b*) tan θ = kot (90° – θ)
tan θ = cot (90° – θ)

(*c*) sek θ = kosek (90° – θ)
sec θ = cosec (90° – θ)

Penyelesaian / *Solution*:



$$\text{Hipotenus} = \sqrt{x^2 + y^2}$$

$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}} \quad ; \quad \text{kos}\,\theta = \frac{x}{\sqrt{x^2+y^2}}$$

$$\tan\theta = \frac{y}{x}$$

(*a*) $$\text{kos}\left(90° - \theta\right) = \frac{y}{\sqrt{x^2+y^2}}$$

$$\sin\theta = \text{kos}\left(90° - \theta\right) \qquad \blacksquare$$

Bersambung

$$\text{kot } A = \frac{1}{\tan A}$$

(*b*) $\text{kot}\left(90^\circ - \theta\right) = 1 \div \frac{x}{y} = \frac{y}{x}$

$\tan\theta = \text{kot}\left(90^\circ - \theta\right)$ ■

$$\text{kosek } A = \frac{1}{\sin A}$$

(*c*) $\text{kosek}\left(90^\circ - \theta\right) = 1 \div \frac{x}{\sqrt{x^2 + y^2}} = \frac{\sqrt{x^2 + y^2}}{x}$

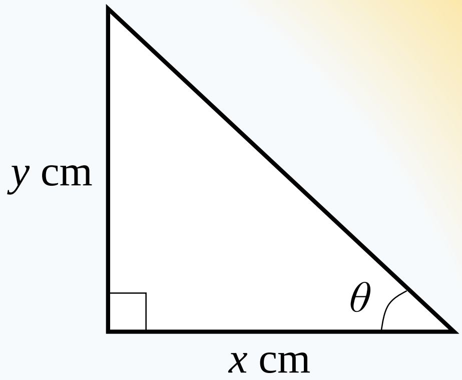
$\text{sek}\,\theta = 1 \div \frac{x}{\sqrt{x^2 + y^2}} = \frac{\sqrt{x^2 + y^2}}{x}$

$$\text{sek } A = \frac{1}{\text{kos } A}$$

$\text{sek}\,\theta = \text{kosek}\left(90^\circ - \theta\right)$ ■

MENERBITKAN IDENTITI ASAS

Rajah menunjukkan satu segi tiga bersudut tegak dengan panjang tapak x cm and tinggi y cm.
Diagram shows a right-angled triangle with a base length of x cm *and height* y cm.



Dengan menggunakan rajah itu, buktikan
By using the diagram, show that

$\sin^2\theta + \text{kos}^2\theta = 1$
$\sin^2\theta + \cos^2\theta = 1$

Penyelesaian / *Solution*:

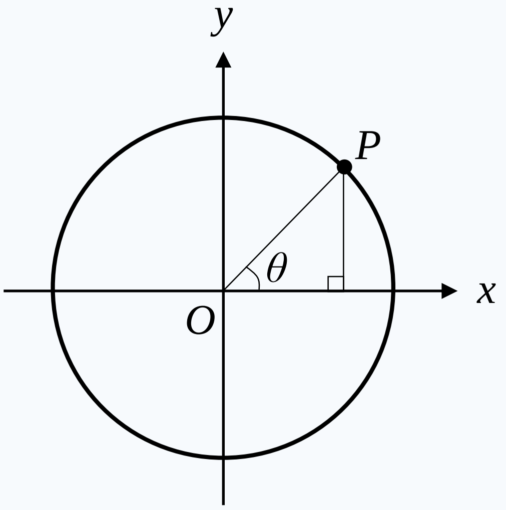
$$\text{Hipotenus} = \sqrt{x^2 + y^2}$$

$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}} \quad ; \quad \text{kos}\,\theta = \frac{x}{\sqrt{x^2+y^2}}$$

$$\begin{aligned}
\text{Sebelah kiri} &= \sin^2\theta + \text{kos}^2\theta \\
&= \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 \\
&= \frac{y^2}{x^2+y^2} + \frac{x^2}{x^2+y^2} \\
&= \frac{x^2+y^2}{x^2+y^2} \\
&= 1 \\
&= \text{Sebelah kanan} \qquad \blacksquare
\end{aligned}$$

MENERBITKAN IDENTITI ASAS

Rajah menunjukkan satu bulatan unit. P ialah titik pada lilitan bulatan itu.
Diagram shows a unit circle. P is a point on the circumference of the circle.



(*a*) Nyatakan koordinat P dalam sebutan θ.
State the coordinates of P in terms of θ.

(*b*) Seterusnya, tunjukkan $\sin^2\theta + \text{kos}^2\theta = 1$.
Hence, show that $\sin^2\theta + \cos^2\theta = 1$.

Penyelesaian / *Solution*:

(*a*) $P(\text{kos}\,\theta, \sin\theta)$

Teorem Pythagoras
Pythagoras' Theorem

(*b*)

$$x^2 + y^2 = 1^2$$

Jejari bulatan unit ialah 1 unit
The radius of a unit circle is 1 unit

$$(\text{kos}\,\theta)^2 + (\sin\theta)^2 = 1$$

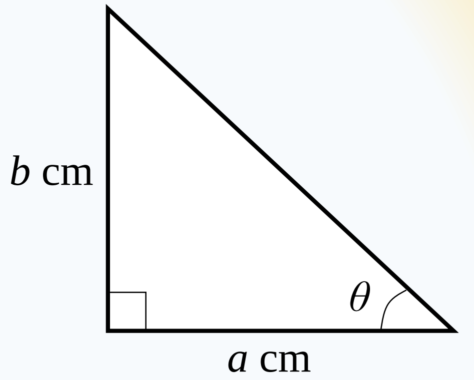
$$\sin^2\theta + \text{kos}^2\theta = 1 \qquad \blacksquare$$

MENERBITKAN IDENTITI ASAS

Rajah menunjukkan satu segi tiga bersudut tegak dengan panjang tapak a cm, tinggi b cm dan hipotenus c cm.
Diagram shows a right-angled triangle with a base length of a cm *and height* b cm *and hypotenuse* c cm.

Dengan menggunakan teorem Pythagoras, buktikan
By using the Pythagoras' theorem, show that

$\sin^2\theta + \text{kos}^2\theta = 1$
$\sin^2\theta + \cos^2\theta = 1$



Penyelesaian / *Solution*:

$$a^2 + b^2 = c^2$$

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2}$$

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

$$(\text{kos}\,\theta)^2 + (\sin\theta)^2 = 1$$

$$\sin^2\theta + \text{kos}^2\theta = 1 \quad \blacksquare$$

MENERBITKAN IDENTITI ASAS

Pertimbangkan identiti trigonometri $\sin^2\theta + \text{kos}^2\theta = 1$.
Consider the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$.

Buktikan
Prove that

(*a*) $1 + \tan^2\theta = \text{sek}^2\theta$
$1 + \tan^2\theta = \sec^2\theta$

(*b*) $1 + \text{kot}^2\theta = \text{kosek}^2\theta$
$1 + \cot^2\theta = \text{cosec}^2\theta$

Penyelesaian / *Solution*:

(*a*)

$$\sin^2\theta + \text{kos}^2\theta = 1$$
$$\frac{\sin^2\theta}{\text{kos}^2\theta} + \frac{\text{kos}^2\theta}{\text{kos}^2\theta} = \frac{1}{\text{kos}^2\theta}$$
$$\tan^2\theta + 1 = \text{sek}^2\theta$$
$$1 + \tan^2\theta = \text{sek}^2\theta \qquad \blacksquare$$

(*b*)

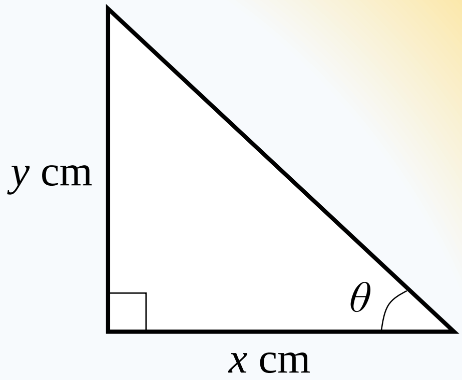
$$\sin^2\theta + \text{kos}^2\theta = 1$$
$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\text{kos}^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$
$$1 + \text{kot}^2\theta = \text{kosek}^2\theta \qquad \blacksquare$$

KAITAN tan (θ + 90°) DENGAN tan θ

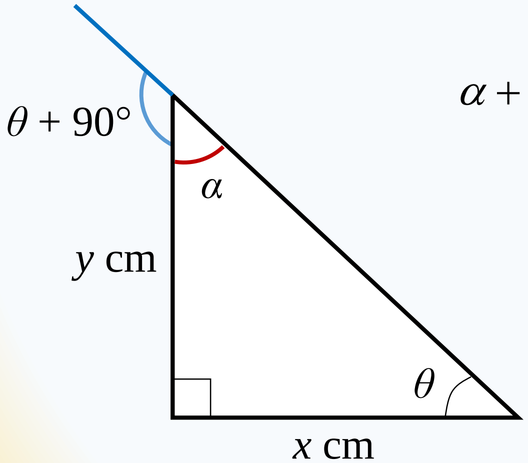
Rajah menunjukkan satu segi tiga bersudut tegak dengan panjang tapak x cm and tinggi y cm.
Diagram shows a right-angled triangle with a base length of x cm *and height* y cm.

Dengan menggunakan rajah itu, buktikan
By using the diagram, show that

$$\tan(\theta + 90°) = -\frac{1}{\tan\theta}$$



Penyelesaian / *Solution*:



$$\alpha + \theta + 90° = 180° \quad ; \quad \tan\theta = \frac{y}{x} \quad ; \quad \tan\alpha = \frac{x}{y}$$

$$\tan(\theta + 90°) = -\tan\alpha$$

$$\tan(\theta + 90°) = -\frac{x}{y}$$

$$\tan(\theta + 90°) = -\left(1 \div \frac{y}{x}\right)$$

$$\tan(\theta + 90°) = -\frac{1}{\tan\theta} \qquad \blacksquare$$

tan (θ + 90°) berada dalam sukuan kedua
tan (θ + 90°) *is located in the second quadrant*

MENERBITKAN RUMUS SUDUT BERGANDA

Tunjukkan kaedah setiap rumus sudut berganda berikut diterbitkan.
Show the method for each of the following double angle formula is derived.

$$\begin{array}{l} \sin 2\theta = 2\sin\theta \operatorname{kos}\theta \\ \sin 2\theta = 2\sin\theta\cos\theta \end{array} \quad ; \quad \begin{array}{l} \operatorname{kos} 2\theta = \operatorname{kos}^2\theta - \sin^2\theta \\ \cos 2\theta = \cos^2\theta - \sin^2\theta \end{array} \quad ; \quad \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

Penyelesaian / *Solution*:

Pertimbangkan $\sin(A+B) = \sin A \operatorname{kos} B + \sin B \operatorname{kos} A$

Gantikan θ dalam A dan B

$$\sin(\theta+\theta) = \sin\theta \operatorname{kos}\theta + \sin\theta \operatorname{kos}\theta$$

$$\sin 2\theta = 2\sin\theta \operatorname{kos}\theta \qquad \blacksquare$$

Pertimbangkan $\operatorname{kos}(A+B) = \operatorname{kos} A \operatorname{kos} B - \sin A \sin B$

Gantikan θ dalam A dan B

$$\operatorname{kos}(\theta+\theta) = \operatorname{kos}\theta \operatorname{kos}\theta - \sin\theta\sin\theta$$

$$\operatorname{kos} 2\theta = \operatorname{kos}^2\theta - \sin^2\theta \qquad \blacksquare$$

Pertimbangkan $\tan(A+B) = \dfrac{\tan A + \tan B}{1-\tan A \tan B}$

Gantikan θ dalam A dan B

$$\tan(\theta+\theta) = \frac{\tan\theta + \tan\theta}{1-\tan\theta\tan\theta}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \qquad \blacksquare$$

MENERBITKAN RUMUS SUDUT BERGANDA

Tunjukkan kaedah setiap rumus sudut berganda berikut diterbitkan.
Show the method for each of the following double angle formula is derived.

$$\text{kos}\,2\theta = \text{kos}^2\theta - \sin^2\theta \quad ; \quad \text{kos}\,2\theta = 2\,\text{kos}^2\theta - 1 \quad ; \quad \text{kos}\,2\theta = 1 - 2\sin^2\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \quad ; \quad \cos 2\theta = 2\cos^2\theta - 1 \quad ; \quad \cos 2\theta = 1 - 2\sin^2\theta$$

Penyelesaian / *Solution*:

Pertimbangkan $\text{kos}(A+B) = \text{kos}\,A\,\text{kos}\,B - \sin A \sin B$

Gantikan θ dalam A dan B

$$\text{kos}(\theta+\theta) = \text{kos}\,\theta\,\text{kos}\,\theta - \sin\theta\sin\theta$$

$$\text{kos}\,2\theta = \text{kos}^2\theta - \sin^2\theta \qquad \blacksquare$$

Pertimbangkan $\sin^2\theta + \text{kos}^2\theta = 1$

$s^2 = 1 - c^2$

$$\text{kos}\,2\theta = \text{kos}^2\theta - \left(1 - \text{kos}^2\theta\right)$$

$$= \text{kos}^2\theta - 1 + \text{kos}^2\theta$$

$$\text{kos}\,2\theta = 2\,\text{kos}^2\theta - 1 \qquad \blacksquare$$

$c^2 = 1 - s^2$

$$\text{kos}\,2\theta = \left(1 - \sin^2\theta\right) - \sin^2\theta$$

$$\text{kos}\,2\theta = 1 - 2\sin^2\theta \qquad \blacksquare$$